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Instantons and the monopole-like equations in eight dimensions

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Abstract: We search for an abelian description of the Yang-Mills instantons on certain eight dimensional manifolds with the special holonomies Spin(7) and SU(4). By mimicing the Seiberg-Witten theory in four dimensions, we propose a set of monopole-like equations governing the 8-dimensional U(1) connections and spinors, which are supposed to be the dual theory of the nonabelian instantons. We also give a naive test of the generalized S-duality in the abelian sector of 8-dimensional Yang-Mills theory. Some problems in this approach are pointed out.

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1. Introduction

Yang-Mills instantons are among the simplest class of BPS states in the low-energy limit of superstring theory. When strings are compactified, it is often important to consider instantons on some special manifolds of dimension other than four. Mathematically, such instantons arise naturally as solutions to the eigenequations of a certain star operator acting on two-forms, and just as in the 4-dimensional case, the Yang-Mills action will reach its minimal values at these solutions. The present paper will be devoted to a study of instantons in eight dimensions.

The notion of Yang-Mills instantons in dimension greater than four is rather old and it may date back to the middle of the 1980's [1, 2]. This problem has raised some renewed interest in recent years as a meanings of generalizing the Donaldson-Witten theory to higher dimensions [3]. In particular, it is quite interesting to see whether Donaldson invariants have the holomorphic extension to Calabi-Yau four folds. Motivated by this as well as by the potential relevance to M-theory and D-brane physics, various aspects of higher dimensional cohomological Yang-Mills theories have been investigated, see e.g. [4]–[7]. An extensive study of the relevant moduli geometry and its relations to certain calibrated submanifolds can be found in ref. [8]. It is expected that the instanton configurations should correspond to supersymmetric D-branes embedded in some manifolds of special holonomies [9]. In a more recent paper, Mariño, Minasian, Moore and Strominger [10] explicitly found that a nonlinear deformation of the higher dimensional instanton equations can be derived from D-branes wrapping around supersymetric cycles, with the deformation parameter characterized by the B-field.

Presumably, the field theoretic approach to the instanton moduli problem based on BRST cohomology [4, 5, 7] is perturbative in nature. The quantum degrees of freedom consist mainly of the nonabelian gauge fields A, which should be considered as fundamental fields when we try to develop a perturbative expansion in terms of the gauge coupling constant. One may ask whether there exists a nonperturbative theory within which one can use collective field variables to explore the underlying strong coupling physics [4]. Inspired by the work of Seiberg and Witten [11] in four dimensions, we tentatively expect that such a theory, if exists, should be closely related to a kind of S-duality. Moreover, in the dual description the collective variables should consist of an abelian gauge field together with a complex spinor satisfying certain "master equations" [12].

In this paper we take a modest step toward an S-dual description for the Yang-Mills instantons on some eight dimensional manifolds with special holonomy groups. Our description mimics the Seiberg-Witten theory [12], in which the nonabelian (anti) self-duality equation $F^+ = 0$ will be replaced by an abelian one, $F^+ = \mathcal{Q}(\psi^\dagger, \psi)$, with ψ being a spinor field obeying the massless Dirac equation and $\mathcal{Q}$ a suitable quadratic form. In writing down the explicit monopole-like equations, we shall consider two types of manifolds: Joyce manifolds [13] of holonomy Spin(7) as well as Calabi-Yau four-folds of holonomy SU(4). We will compare our equations with the monopole equations constructed in 4-dimensions, and point out some problems yet to be resolved.

As a physical motivation of this investigation, we will also discuss the free abelian sector embedded in 8-dimensional Yang-Mills theory and provide a naive path-integral test of the S-duality in that sector. This discussion is an eight dimensional generalization of the usual electric-magnetic duality in four dimensions. The duality structure in eight dimensions may be alternatively understood as the existence of different gauge-fixings of a topological symmetry [14].

The paper is organized as follows. In section 2 we recall some known facts about Yang-Mills instantons on manifolds of holonomy groups Spin(7) and SU(4). In section 3, we give an explicit construction of the monopole-like equations. In section 4 we turn to a discussion of the generalized S-duality in 8-dimensional abelian gauge theory. Finally we provide an appendix where some useful properties of the 8-dimensional Clifford algebra are presented.

2. Yang-Mills instantons in eight dimensions

Yang-Mills instantons in eight dimensions [1, 2, 4] originate from a generalization of the usual concept of (anti) self-duality. Suppose that we have an eight-dimensional riemannian manifold X on which a closed 4-form Ω is defined. One can use this Ω to construct a star operator $*_\Omega : \Lambda^2 \to \Lambda^2$, $*_\Omega F \equiv *(\Omega \wedge F)$ acting on the space of two-forms. The (anti) self-duality equations are then formulated as the eigenequation $*_\Omega F = \lambda F$ of the star operator.

In terms of components, the action of $*_\Omega$ is given by

$$(*_\Omega F)_{\mu\nu} = \frac{1}{2}\Omega_{\mu\nu\alpha\beta}F^{\alpha\beta}\,. \tag{2.1}$$

Thus, if Ω obeys an identity of the form

$$\Omega_{\mu\nu\alpha\beta}\Omega^{\alpha\beta\sigma\tau} = A(\delta^\sigma_\mu\delta^\tau_\nu - \delta^\tau_\mu\delta^\sigma_\nu) + B\Omega_{\mu\nu}{}^{\sigma\tau}\,, \tag{2.2}$$

with some real constants A and B (where $A > 0$ depends on the normalization of Ω), then the eigenequation has two solutions $\lambda = \lambda^\pm$, $F = F^\pm$, determined by:

$$\begin{aligned}\lambda^\pm &= \frac{B \mp \sqrt{B^2+8A}}{4}\,,\\ F^\pm_{\mu\nu} &= \pm\frac{2}{\sqrt{B^2+8A}}\left(\lambda^\mp F_{\mu\nu} - \frac{1}{2}\Omega_{\mu\nu\alpha\beta}F^{\alpha\beta}\right).\end{aligned} \tag{2.3}$$

One may easily verify that $F_{\mu\nu} = F^+_{\mu\nu} + F^-_{\mu\nu}$. Accordingly, the space Λ^2 of two-forms is decomposed into a direct sum $\Lambda^2_+ \oplus \Lambda^2_-$ of the eigenspaces of $*_\Omega$. We will call F to be a self-dual form (resp. anti self-dual form) if it belongs to Λ^2_- (resp. Λ^2_+). The condition that F is self-dual can be simply written as $F^+_{\mu\nu} = 0$.

Actually we shall forcus on some G-bundle $E \to X$ and consider its connections A. In this context, A is called a self-dual instanton (upto gauge transformations) if the corresponding curvature two-form $F(A)$ obeys the self-duality equation $F^+(A) = 0$. An instanton will minimize the Yang-Mills action functional

$$S_{YM}[A] = \frac{1}{2g^2}\int_X d^8x\sqrt{g}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} \equiv \frac{1}{2g^2}||F||^2\,. \tag{2.4}$$

In fact, if we decompose F into components $F^\pm \in \Lambda^2_\pm$, then (2.3) gives:

$$S_{YM}[A] = \frac{1}{g^2(B+\sqrt{B^2+8A})}\left(2\int_X \Omega\wedge\mathrm{Tr}(F\wedge F) + \sqrt{B^2+8A}||F^+||^2\right). \tag{2.5}$$

So the Yang-Mills action can be written as a non-negative term proportional to $||F^+||^2$, plus a topological invariant. Clearly, such an action will reach its minimal values at $F^+ = 0$.

Manifolds with Spin(7) **holonomy.** Now we briefly discuss the case when X has the holonomy group Spin(7). This means that X is spin, and there is a real, non-zero parallel spinor $\zeta \in S^+$ on X invariant under the action of Spin(7) $\subset$ Spin(8). We will normalize such a spinor by imposing the condition

$$\zeta^T\zeta = 1\,. \tag{2.6}$$

According to the standard isomorphism $S^+ \otimes_S S^+ \cong \Lambda^0 \oplus \Lambda^4_+$, $S^+ \wedge S^+ \cong \Lambda^2$ between the space of forms and tensor product of the Clifford module [15], the 4-form Ω considered above can be constructed as a "bispinor"

$$\Omega_{\mu\nu\alpha\beta} = \zeta^T \Gamma_{\mu\nu\alpha\beta} \zeta \,, \tag{2.7}$$

where $\Gamma_{\mu\nu\alpha\beta}$ denotes the anti-symmetrized 4-fold product of the γ-matrices Γ_μ in 8 dimensions, with a prefactor 1/4! included. Our convention of choosing the γ-matrices is given in the appendix.

Eq. (2.7) obviously defines a Spin(7) invariant rank-4 tensor. This tensor enjoys a couple of useful properties: first, it is covariantly constant, so that Ω gives rise to a closed form. Second, using the γ-matrix identity $\Gamma_{\mu\nu\alpha\beta}\Gamma_9 = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta\lambda\rho\sigma\tau}\Gamma^{\lambda\rho\sigma\tau}$, one easily sees that Ω is self-dual with respect to the usual Hodge star operator, namely $*\Omega = \Omega$, — this agrees with the fact that the symmetric tensor product of S^+ contains Λ^4_+. The final property which we shall use is that (2.7) obeys an identity [7] of the form (2.2):

$$\Omega_{\mu\nu\alpha\beta}\Omega^{\alpha\beta\sigma\tau} = 6(\delta^\sigma_\mu\delta^\tau_\nu - \delta^\tau_\mu\delta^\sigma_\nu) - 4\Omega_{\mu\nu}{}^{\sigma\tau} \,. \tag{2.8}$$

In particular Ω is normalized to be $||\Omega||^2 \equiv \frac{1}{4!}\Omega_{\mu\nu\alpha\beta}\Omega^{\mu\nu\alpha\beta} = 14$.

Hence, on a manifold X of holonomy Spin(7), the (anti) self-duality equation $*_\Omega F = \lambda F$ has eigenvalues $\lambda^+ = -3$, $\lambda^- = 1$; the curvature two-form $F_{\mu\nu} = F^+_{\mu\nu} + F^-_{\mu\nu}$ is decomposed orthogonally into an anti self-dual part $F^+_{\mu\nu}$ and a self-dual part $F^-_{\mu\nu}$, with

$$\begin{aligned} F^+_{\mu\nu} &= \frac{1}{4}\left(F_{\mu\nu} - \frac{1}{2}\Omega_{\mu\nu\alpha\beta}F^{\alpha\beta}\right) \in \Lambda^2_+, \\ F^-_{\mu\nu} &= \frac{1}{4}\left(3F_{\mu\nu} + \frac{1}{2}\Omega_{\mu\nu\alpha\beta}F^{\alpha\beta}\right) \in \Lambda^2_- \,. \end{aligned} \tag{2.9}$$

The Yang-Mills instantons are thus described by the equation $F^+_{\mu\nu} = 0$. The dimensions of $\Lambda^2_\pm$ can be determined by a group-theoretic consideration [4], and the result turns out to be

$$\dim \Lambda^2_+ = 7 \,, \qquad \dim \Lambda^2_- = 21 \,. \tag{2.10}$$

For an alternative derivation of this result, note that $\mathrm{Tr}(*_\Omega) = 0$, so $(-3)\cdot \dim \Lambda^2_+ + 1 \cdot \dim \Lambda^2_- = 0$. This together with $\dim \Lambda^2_+ + \dim \Lambda^2_- = \dim \Lambda^2 = 28$ gives (2.10).

Manifolds with SU(4) **holonomy.** As just mentioned, X has holonomy Spin(7) if there is a generic parallel spinor $\zeta \neq 0$ defined on it. However, this holonomy group may reduce to a subgroup of Spin(7) when the parallel spinor obeys certain particular conditions [15]. For example, one has the holonomy reduction Spin(7) $\to$ SU(4) provided there exists a parallel *pure spinor* ζ on X. Here we shall describe in some detail what a pure spinor is and explain why the existence of such a spinor will cause the manifold to have holonomy SU(4) [15]. We will then discuss a holomorphic version of the Yang-Mills instanton equations [4].

Let $\mathrm{Cl}_c(8) = \mathrm{Cl}(8) \otimes \mathbb{C}$ be the 8-dimensional Clifford algebra over $\mathbb{C}$, and let S_c be a complex spinor space, on which an irreducible representation ρ^c of $\mathrm{Cl}_c(8)$ is defined. Each spinor $\psi \in S_c$ can be associated to a $\mathbb{C}$-linear map

$$f_\psi : \mathbb{C}^8 \to S_c\,, \qquad f_\psi(u) = \rho^c(u) \cdot \psi\,, \tag{2.11}$$

where u is a complex linear combination of the Clifford generators $e_\mu \in \mathrm{Cl}(8)$ and we have identified the space of all such linear combinations with $\mathbb{C}^8$. Let us consider the kernel of this map in the case $\psi \neq 0$. If $u = a^\mu e_\mu$, $v = b^\mu e_\mu \in \mathrm{Ker}\, f_\psi$ (with complex coefficients a^μ, b^μ), then $\rho^c(u)\psi = \rho^c(v)\psi = 0 \Rightarrow 0 = \rho^c(\{u,v\})\psi = -2g_{\mu\nu}a^\mu b^\nu \psi$. It follows that the space $\mathrm{Ker}\, f_\psi$ is orthogonal to its complex conjugate $\overline{\mathrm{Ker}\, f_\psi}$ with respect to the standard hermitean inner product $(a^\mu e_\mu, b^\nu e_\nu) \equiv \langle a^\mu e_\mu, \bar{b}^\nu e_\nu\rangle = g_{\mu\nu} a^\mu \bar{b}^\nu$ on $\mathbb{C}^8$. Thus, since $\mathrm{Ker}\, f_\psi \oplus \overline{\mathrm{Ker}\, f_\psi} \subset \mathbb{C}^8$, we see that the complex dimensions of $\mathrm{Ker}\, f_\psi$ should not exceed 4. By definition, we say that ψ is a pure spinor if $\dim_c \mathrm{Ker}\, f_\psi$ reaches its maximally allowed value 4.

We will take the complex spinor space to be the complexification of the real $\mathrm{Cl}(8)$-module $S \cong \mathbb{R}^{16}$: $S_c = S \otimes \mathbb{C}$. Spinors in such a space can be written as linear combinations of a basis of S with complex coefficients. Also, one takes ρ^c to be the $\mathbb{C}$-linear extension of the γ-matrix representation of $\mathrm{Cl}(8)$. This allows us to choose $\Gamma_\mu \equiv \rho^c(e_\mu)$ as given in the appendix.

Since $\Gamma_9 = I \oplus (-I)$, S_c is decomposed into subspaces $S_c^\pm$ of positive and negative chiralities. By construction, $\psi \in S_c^+$ means that ψ is a linear combination of some real spinors in S^+ with complex coefficients, so its complex conjugate $\bar{\psi}$ also has positive chirality. One can show that [15] if ψ is a pure spinor, then either ψ will be entirely in S_c^+ or it will be entirely in S_c^-; namely, ψ has a definite chirality. To see this, note that a change in the orthonormal basis $\{e_\mu\}$ of $\mathbb{R}^8 \subset \mathrm{Cl}(8)$ will leave the matrix $\Gamma_9 = \rho^c(e_1 \cdots e_8)$ invariant, upto a factor ± 1 depending on the relative ordering. Also note that if ψ is a pure spinor, then $\mathrm{Ker}\, f_\psi \oplus \overline{\mathrm{Ker}\, f_\psi} = \mathbb{C}^8$, so any basis $\{h_{\bar{i}}, 1 \leq \bar{i} \leq 4\}$ of $\mathrm{Ker}\, f_\psi$ along with its complex conjugate $\{h_i\} \subset \overline{\mathrm{Ker}\, f_\psi}$ provides a basis of $\mathbb{C}^8$. We can take $\{h_{\bar{i}}\}$ to be orthonormal with respect to the natural hermitean metric on $\mathbb{C}^8$. Then the following vectors

$$\begin{aligned}
&e_1 = \frac{1}{\sqrt{2}}(h_1 + h_{\bar{1}}), \quad e_2 = \frac{1}{\sqrt{2}i}(h_1 - h_{\bar{1}}), \quad e_3 = \frac{1}{\sqrt{2}}(h_2 + h_{\bar{2}}), \quad e_4 = \frac{1}{\sqrt{2}i}(h_2 - h_{\bar{2}})\\
&e_5 = \frac{1}{\sqrt{2}}(h_3 + h_{\bar{3}}), \quad e_6 = \frac{1}{\sqrt{2}i}(h_3 - h_{\bar{3}}), \quad e_7 = \frac{1}{\sqrt{2}}(h_4 + h_{\bar{4}}), \quad e_8 = \frac{1}{\sqrt{2}i}(h_4 - h_{\bar{4}})
\end{aligned} \tag{2.12}$$

form an orthonormal basis of $\mathbb{R}^8$, as they are all invariant under complex conjugation. Now as $h_{\bar{1}} \in \mathrm{Ker}\, f_\psi$, we have $\rho^c(e_1 - ie_2)\psi = 0 \Rightarrow \rho^c(e_1^2 - ie_1e_2)\psi = 0 \Rightarrow \rho^c(e_1e_2)\psi = i\psi$. Similar arguments lead to $\rho^c(e_3e_4)\psi = \rho^c(e_5e_6)\psi = \rho^c(e_7e_8)\psi = i\psi$. Thus we find $\Gamma_9\psi = \psi$, indicating that ψ has the positive chirality. A choice of the basis with different ordering will give $\Gamma_9\psi = -\psi$, but in any case the chirality of a pure spinor is definite.

Given now a pure spinor $\zeta \in S_c^+$, let us consider the maximal subgroup of Spin(8) that keeps ζ invariant. An element of Spin(8) can act adjointly on the real vector space spanned by $\{e_\mu\}$,

$$e_\mu \to e'_\mu = g^{-1} e_\mu g \equiv \rho_v(g)_\mu{}^\nu e_\nu \,, \qquad \rho_v(g) \in \mathrm{SO}(8)\,, \tag{2.13}$$

which defines the representation $\mathbf{8}_v$ of Spin(8). The action (2.13) may be viewed as a change in the basis and it clearly preserves both the orthonormal property and the ordering of the basis. The $\mathbb{C}$-linear extension of ρ_v, which we will denote by ρ_v^c, is defined naturally on the space $\mathbb{C}^8 = \mathrm{Ker}\, f_\zeta \oplus \overline{\mathrm{Ker}\, f_\zeta}$. For generic $g \in \mathrm{Spin}(8)$, neither the subspace $\mathrm{Ker}\, f_\zeta$ nor $\overline{\mathrm{Ker}\, f_\zeta}$ is invariant under the action of $\rho_v^c(g)$. Elements of SO(8) that leaves these subspaces invariant will map one orthonormal basis $\{h_i\} \subset \overline{\mathrm{Ker}\, f_\zeta}$ (and $\{h_{\bar{i}}\} \subset \mathrm{Ker}\, f_\zeta$) into another, thus forming the subgroup SU(4). Such elements arise from those $g \in \mathrm{Spin}(8)$ keeping ζ invariant. Indeed, for any $h \in \mathrm{Ker}\, f_\zeta$ and $\rho^c(g)\zeta = \zeta$, we have $\rho^c(g^{-1}hg)\zeta = \rho^c(g)^{-1}\rho^c(h)\rho^c(g)\zeta = \rho^c(g)^{-1}\rho^c(h)\zeta = 0 \Rightarrow g^{-1}hg \in \mathrm{Ker}\, f_\zeta$. We thus conclude that the isotropy group of a pure spinor $\zeta \in S_c^+$ is SU(4). A globalized version of this discussion leads to the statement [15]:

There exists a parallel pure spinor on $X \Longleftrightarrow X$ has the holonomy group SU(4) *(or its subgroup).*

Now we take a pure spinor $\zeta \in S_c^+$ and fix the almost complex structure on $\mathbb{R}^8$ as in (2.12), so that the basis h_i of $\overline{\mathrm{Ker}\, f_\zeta}$ and the basis $h_{\bar{i}}$ of $\mathrm{Ker}\, f_\zeta$ have the γ-matrix representation:

$$\gamma_1 = \frac{1}{\sqrt{2}}(\Gamma_1 + i\Gamma_2)\,,\ \gamma_2 = \frac{1}{\sqrt{2}}(\Gamma_3 + i\Gamma_3)\,,\ \gamma_3 = \frac{1}{\sqrt{2}}(\Gamma_5 + i\Gamma_6)\,,\ \gamma_4 = \frac{1}{\sqrt{2}}(\Gamma_7 + i\Gamma_8) \tag{2.14}$$

$$\gamma_{\bar{1}} = \frac{1}{\sqrt{2}}(\Gamma_1 - i\Gamma_2)\,,\ \gamma_{\bar{2}} = \frac{1}{\sqrt{2}}(\Gamma_3 - i\Gamma_3)\,,\ \gamma_{\bar{3}} = \frac{1}{\sqrt{2}}(\Gamma_5 - i\Gamma_6)\,,\ \gamma_{\bar{4}} = \frac{1}{\sqrt{2}}(\Gamma_7 - i\Gamma_8)\,. \tag{2.15}$$

The complex Clifford algebra is determined by the relations

$$\gamma_i\gamma_j + \gamma_j\gamma_i = \gamma_{\bar{i}}\gamma_{\bar{j}} + \gamma_{\bar{j}}\gamma_{\bar{i}} = 0\,, \qquad \gamma_i\gamma_{\bar{j}} + \gamma_{\bar{j}}\gamma_i = -2g_{i\bar{j}}\,. \tag{2.16}$$

We also need the dual basis

$$\gamma^i = g^{i\bar{j}}\gamma_{\bar{j}}, \quad \gamma^{\bar{i}} = \gamma_j g^{j\bar{i}} \tag{2.17}$$

as well as their anti-symmetrized products $\gamma^{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$. With this notation, a (p,q)-form $t \in \Lambda^{p,q}$ has a natural representation in terms of the γ-matrices:

$$t \leftrightarrow \frac{1}{p!q!} t_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\gamma^{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\,. \tag{2.18}$$

Moreover, each such form should be associated to a bispinor $\phi^\dagger\gamma_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\psi \in S_c \otimes S_c$ as in the real case. Note that the isomorphism [15] between the tensor product of spinors and forms has a $\mathbb{C}$-bilinear extension to the complex case

$$\rho^c \otimes \rho^c \cong 2(1 + \rho_v^c + \wedge^2\rho_v^c + \wedge^3\rho_v^c) + \wedge^4\rho_v^c\,, \tag{2.19}$$

which shows that $(S_c^+ \oplus S_c^-) \otimes (S_c^+ \oplus S_c^-)$ can be identified with forms in $\wedge^*\mathbb{C}^8$.

To warm up the complex Clifford calculus, let us establish an isomorphism between $S_c^\pm \otimes \zeta^\dagger$ and certain particular forms. Since $\gamma_{\bar{i}}$ is in $\mathrm{Ker} f_\zeta$, we have

$$\gamma_{\bar{i}}\zeta = \gamma^i \zeta = 0 \quad \Rightarrow \quad \zeta^\dagger \gamma_i = \zeta^\dagger \gamma^{\bar{i}} = 0 \,, \tag{2.20}$$

and this gives to $\zeta^\dagger \gamma_i \gamma_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} = 0$. One may use this and the γ-matrix identity

$$\gamma_i \gamma_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} = \gamma_{i i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} + \sum_{k=1}^{q} (-1)^{k+p} g_{i\bar{j}_k} \gamma_{i_1 \cdots i_p \bar{j}_1 \cdots \hat{\bar{j}}_k \cdots \bar{j}_q} \tag{2.21}$$

to deduce that the $(p{+}1, q)$ type bispinor $\zeta^\dagger \gamma_{i i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \psi$ is in fact a linear combination of some $(p, q-1)$ forms. This process can be proceeded inductively and we find that the $(p{+}1, q)$-bispinor finally becomes a linear combination of $\zeta^\dagger \gamma_{\bar{j}_1 \cdots \bar{j}_{q-p-1}} \psi$ if $q \geq p{+}1$ or a linear combination of $\zeta^\dagger \gamma_{i_1 \cdots i_{p+1-q}} \psi \equiv 0$ if $q < p+1$. Accordingly, for arbitrary $\psi \in S_c = S_c^+ \oplus S_c^-$, the tensor product $\zeta^\dagger \otimes \psi$ can be identified to a form in $\Lambda^{0,*}$. Note that with our convention of the γ-matrices, $\gamma_{\bar{j}_1 \cdots \bar{j}_q}$ is block diagonal for $q =$ even and off-diagonal for $q =$ odd. It follows that

$$S_c^+ \otimes \mathbb{C} \cong \Lambda^{0,\mathrm{even}} \,, \qquad S_c^- \otimes \mathbb{C} \cong \Lambda^{0,\mathrm{odd}} \tag{2.22}$$

here $\mathbb{C}$ is the complex 1-dimensional space generated by $\zeta^\dagger$. Similarly, tensor products $\psi_\pm \otimes \zeta$ for $\psi_\pm \in S_c^\pm$ should be identified with a form in $\Lambda^{\mathrm{even},0}$ and in $\Lambda^{\mathrm{odd},0}$, respectively.

Now we give a suitable normalization of ζ. In the Spin(7) case we have simply imposed the condition $\zeta^T \zeta = 1$. However, this normalization condition cannot be adopted here for a pure spinor ζ. In fact from (2.20)–(2.21) we see that $\gamma_{i\bar{j}}\zeta = g_{i\bar{j}}\zeta$, so that $\zeta^T \gamma_{i\bar{j}} \zeta = g_{i\bar{j}}(\zeta^T \zeta)$, which together with the anti-symmetric property of the matrix $\gamma_{i\bar{j}}$ implies $\zeta^T \zeta = 0$. Nevertheless, one can still impose another normalization condition

$$\zeta^\dagger \zeta = 1 \,, \tag{2.23}$$

and this looks more natural when we work in complex spaces. Using this normlization, we define an SU(4) invariant closed $(4,0)$-form Ω with the components

$$\Omega_{ijkl} = \zeta^T \gamma_{ijkl} \zeta \,. \tag{2.24}$$

Some properties of (2.24) can be explored using a complex version of the Fierz rearrangement formula:

$$\zeta \zeta^T = \frac{1}{16 \cdot 4!} \Omega_{ijkl} \gamma^{ijkl} \,, \tag{2.25}$$

$$\zeta \zeta^\dagger = \frac{1}{16}(1 + g^{i\bar{j}} \gamma_{i\bar{j}})(1 + \Gamma_9) + \frac{1}{32} g^{i\bar{l}} g^{j\bar{k}} \gamma_{ij\bar{k}\bar{l}} \,. \tag{2.26}$$

For example one may apply (2.25) to a quick computation of the norm $||\Omega||^2$. By definition, $||\Omega||^2 \equiv \frac{1}{4!}\Omega_{ijkl}\bar{\Omega}^{ijkl}$, $\bar{\Omega}^{ijkl} \equiv g^{i\bar{i}} g^{j\bar{j}} g^{k\bar{k}} g^{l\bar{l}} \bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}}$, where $\bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}} \in \Lambda^{0,4}$ is the complex conjugate of Ω. Since $\bar{\Omega}^{ijkl} = \zeta^\dagger \gamma^{ijkl} \bar{\zeta}$, we see that (2.25) gives $||\Omega||^2 = 16$. Notice that this normalization of Ω is different from the Spin(7) case, where $||\Omega||^2 = 14$. As an application of (2.26), one can establish a more useful identity

$$\Omega_{ijkl} \bar{\Omega}^{mnkl} = 32(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m) \,, \tag{2.27}$$

which takes a form similar to (2.2).

We turn now to the self-duality equations. Given the SU(4) invariant (4,0)-form Ω_{ijkl} defined as above, its complex conjugate $\bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}}$, a (0,4)-form, may be used to construct an anti-linear star operator $*_\Omega : \Lambda^{0,2} \to \Lambda^{0,2}$ by means of

$$(*_\Omega \beta)_{\bar{i}\bar{j}} = \frac{1}{2} \bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}} \bar{\beta}^{\bar{k}\bar{l}} \,, \qquad \forall \beta_{\bar{i}\bar{j}} \in \Lambda^{0,2} \,, \tag{2.28}$$

where $\bar{\beta}_{ij} \in \Lambda^{2,0}$ denotes the complex conjugate of $\beta_{\bar{i}\bar{j}}$ and $\bar{\beta}^{\bar{k}\bar{l}} \equiv g^{i\bar{k}} g^{j\bar{l}} \bar{\beta}_{ij}$. Thus, if $F \in \Lambda^2$ is a curvature 2-form, we can decompose it into $F = F^{(2,0)} + F^{(1,1)} + F^{(0,2)}$ with $F^{(2,0)} = -\overline{F^{(0,2)}}$ (assuming that the connection is unitary), and define

$$(*_\Omega F^{(0,2)})_{\bar{i}\bar{j}} = -\frac{1}{2} \bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}} F^{(2,0)\bar{k}\bar{l}} \,, \qquad (*_\Omega F^{(2,0)})_{ij} = -\frac{1}{2} \Omega_{ijkl} F^{(0,2)kl} \,. \tag{2.29}$$

Note that the (1,1)-component of F is intact under the action of $*_\Omega$.

Just as in the Spin(7) case, the (anti) self-duality equations should be formulated as the eigenvalue equation of $*_\Omega$. Hence, in terms of components, we call $F_{\mu\nu}$ to be (anti) self-dual if they satisfy the conditions

$$-\frac{1}{2} \bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}} F^{(2,0)\bar{k}\bar{l}} = \lambda F^{(0,2)}_{\bar{i}\bar{j}} \,, \qquad -\frac{1}{2} \Omega_{ijkl} F^{(0,2)kl} = \lambda F^{(2,0)}_{ij} \,. \tag{2.30}$$

Here the eigenvalues $\lambda \in \mathbb{R}$ are determined by

$$\lambda = \lambda_\pm \,, \qquad \lambda_+ = -4 \,, \qquad \lambda_- = 4 \,. \tag{2.31}$$

Accordingly, the space of (0,2)-forms gets decomposed into the two eigenspaces of $*_\Omega$, $\Lambda^{0,2} = \Lambda^{0,2}_+ \oplus \Lambda^{0,2}_-$, where $\Lambda^{0,2}_\pm$ correspond to the eigenvalues $\lambda_\pm$, respectively. The (0,2) component of $F \in \Lambda^2$ then decomposes into an anti self-dual part $F^{(0,2)}_+ \in \Lambda^{0,2}_+$ and a self-dual part $F^{(0,2)}_- \in \Lambda^{0,2}_-$, with

$$F^{(0,2)}_{\pm \bar{i}\bar{j}} = \frac{1}{2} \left(F^{(0,2)}_{\bar{i}\bar{j}} \pm \frac{1}{8} \bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}} F^{(2,0)\bar{k}\bar{l}} \right) . \tag{2.32}$$

Holomorphic Yang-Mills instantons are thus characterized by the self-duality equation

$$F^{(0,2)}_+(A) = 0 \quad \Longleftrightarrow \quad -\frac{1}{2} \bar{\Omega}_{\bar{i}\bar{j}\bar{k}\bar{l}} F^{(2,0)\bar{k}\bar{l}} = 4 F^{(0,2)}_{\bar{i}\bar{j}} \,. \tag{2.33}$$

Sometimes it is useful to have a more compact description for the $*_\Omega$ operator, without reference to the unitary basis given in (2.12). To give such a description, note that there is a natural hermitean inner product on $\Lambda^{0,q}$: for two arbitrary $(0,q)$ forms $\alpha_{\bar i_1\cdots\bar i_q}$ and $\beta_{\bar i_1\cdots\bar i_q}$, we can define an SU(4) invariant paring

$$\langle\alpha,\beta\rangle \equiv \frac{1}{q!}\alpha_{\bar i_1\cdots\bar i_q}\bar\beta^{\bar i_1\cdots\bar i_q} = \frac{1}{q!} g^{i_1\bar j_1}\cdots g^{i_q\bar j_q}\bar\beta_{i_1\cdots i_q}\alpha_{\bar j_1\cdots\bar j_q}\,, \tag{2.34}$$

which is linear in α and anti-linear in β. In terms of this inner product, one then introduces an operator $*_\Omega : \Lambda^{0,q} \to \Lambda^{0,4-q}$ through [4]

$$\alpha\wedge *_\Omega\beta = \langle\alpha,\beta\rangle\bar\Omega\,. \tag{2.35}$$

Clearly, this description manifests the SU(4) invariance and does not depend on a particular choice of the basis of $\mathrm{Ker} f_\zeta$. One may see that this definition agrees with the previous one for $q = 2$.

Actually, it is possible to consider a slightly generalized case where we have an SU(n) invariant $(n,0)$-form $\Omega_{i_1\cdots i_n}$ defined on some $2n$-dimensional space X. In that case, the star operator constructed by (2.35) should map $\beta\in\Lambda^{0,q}$ into $*_\Omega\beta\in\Lambda^{0,n-q}$, so that $\alpha\wedge *_\Omega\beta$ is a $(0,n)$-form. The component of the left hand side of (2.35) is $\frac{1}{q!(n-q)!}\alpha_{[\bar i_1\cdots\bar i_q}(*_\Omega\beta)_{\bar i_{q+1}\cdots\bar i_n]}$, while the component of the right hand side of (2.35) is $\frac{1}{n!q!}\alpha_{\bar j_1\cdots\bar j_q}\bar\beta^{\bar j_1\cdots\bar j_q}\bar\Omega_{\bar i_1\cdots\bar i_n}$; making them equal to each other for arbitrary $\alpha\in\Lambda^{0,q}$ leads to

$$\delta^{\bar j_1}_{[\bar i_1}\cdots\delta^{\bar j_n}_{\bar i_n]}(*_\Omega\beta)_{\bar j_{q+1}\cdots\bar j_n} = \frac{(n-q)!}{n!}\bar\beta^{\bar j_1\cdots\bar j_q}\bar\Omega_{\bar i_1\cdots\bar i_n}\,.$$

By contracting the q pairs $(\bar i_1,\bar j_1),\ldots,(\bar i_q,\bar j_q)$ of the tensor indices in this equation, and then using the identity

$$\delta^{\bar i_1}_{[\bar i_1}\cdots\delta^{\bar i_q}_{\bar i_q}\delta^{\bar j_{q+1}}_{\bar i_{q+1}}\cdots\delta^{\bar j_n}_{\bar i_n]} = \frac{q!(n-q)!}{n!}\delta^{\bar j_{q+1}}_{[\bar i_{q+1}}\cdots\delta^{\bar j_n}_{\bar i_n]}\,,$$

we see that

$$(*_\Omega\beta)_{\bar i_{q+1}\cdots\bar i_n} = \frac{1}{q!}\bar\beta^{\bar i_1\cdots\bar i_q}\bar\Omega_{\bar i_1\cdots\bar i_q\bar i_{q+1}\cdots\bar i_n} = \frac{(-1)^{q(n-q)}}{q!}\bar\Omega_{\bar i_{q+1}\cdots\bar i_n\bar i_1\cdots\bar i_q}\bar\beta^{\bar i_1\cdots\bar i_q}\,.$$

In particular for $n = 4$ and $q = 2$, this reduces to our earlier definition (2.28).

3. The monopole-like equations

Non-abelian instantons constitute a moduli problem. In 4-dimensions, this problem can be transformed into a simpler problem, where the gauge fields A are taken to be abelian and one introduces certain new degrees of freedom — a spinor ψ, which satisfies the massless Dirac equations $\Gamma_\mu D^\mu_A\psi = 0$. The couplings between A and ψ are described by, in addtion to the Dirac eqautions, a non-linear relation $F^+(A) = \mathcal{Q}(\psi,\bar\psi)$, where $\mathcal{Q}$ is some quadratic form in ψ, taking values in the anti self-dual part Λ^2_+ of two-forms. This is the basic setup of the Seiberg-Witten theory [12]. Now a natural question arises as whether we can find an 8-dimensional analog of such a theory.

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Manifolds with Spin(7) **holonomy.** On 8-dimensional manifold X with Spin(7) holonomy, there also exists a natural quadratic form $\mathcal{Q}(\psi,\bar{\psi})$ valued in Λ^2_+. Indeed, given a complex line bundle $\mathcal{L}$ and a spinor field $\psi \in S^+ \otimes \mathcal{L}$, one can construct a two-form $\zeta^T\Gamma_{\mu\nu}\psi = -\psi^T\Gamma_{\mu\nu}\zeta$ and, according to [7], it takes values in $\Lambda^2_+ \otimes \mathcal{L}$. One can also form the inner product $\bar{\psi}^T\zeta \in \Lambda^0 \otimes \mathcal{L}^{-1}$. It follows that the quadratic form $\mathcal{Q}_{\mu\nu}(\psi,\bar{\psi}) = (\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi)$ belongs to $(\Lambda^0 \otimes \mathcal{L}^{-1}) \otimes (\Lambda^2_+ \otimes \mathcal{L}) \cong \Lambda^2_+ \otimes \mathbb{C}$. Thus, by choosing a unitary connection A of $\mathcal{L}$, it is possible to write down an 8-dimensional analog of the Seiberg-Witten equations

$$\begin{aligned} &F^+_{\mu\nu}(A) = ia\cdot \mathrm{Re}\left[(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi)\right] + ib\cdot \mathrm{Im}\left[(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi)\right], \\ &\Gamma_\mu D^\mu_A\psi = 0\,, \end{aligned} \tag{3.1}$$

where a, b are real constants.

In order to see that both of the real part and the imaginary part of $\mathcal{Q}$ are not necessarily vanishing for generic $\psi \in S^+ \otimes \mathcal{L}$, one may work out $(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi)$ in a fully explicit form. Using the γ-matrices given in the appendix we find

$$\begin{aligned}
\zeta^T\Gamma_{12}\psi &= -\zeta_1\psi_2 + \zeta_2\psi_1 + \zeta_3\psi_4 - \zeta_4\psi_3 - \zeta_5\psi_6 + \zeta_6\psi_5 + \zeta_7\psi_8 - \zeta_8\psi_7 \\
\zeta^T\Gamma_{13}\psi &= \zeta_1\psi_4 - \zeta_2\psi_3 + \zeta_3\psi_2 - \zeta_4\psi_1 + \zeta_5\psi_8 - \zeta_6\psi_7 + \zeta_7\psi_6 - \zeta_8\psi_5 \\
\zeta^T\Gamma_{14}\psi &= -\zeta_1\psi_5 + \zeta_2\psi_6 - \zeta_3\psi_7 + \zeta_4\psi_8 + \zeta_5\psi_1 - \zeta_6\psi_2 + \zeta_7\psi_3 - \zeta_8\psi_4 \\
\zeta^T\Gamma_{15}\psi &= \zeta_1\psi_6 + \zeta_2\psi_5 + \zeta_3\psi_8 + \zeta_4\psi_7 - \zeta_5\psi_2 - \zeta_6\psi_1 - \zeta_7\psi_4 - \zeta_8\psi_3 \\
\zeta^T\Gamma_{16}\psi &= -\zeta_1\psi_3 - \zeta_2\psi_4 + \zeta_3\psi_1 + \zeta_4\psi_2 + \zeta_5\psi_7 + \zeta_6\psi_8 - \zeta_7\psi_5 - \zeta_8\psi_6 \\
\zeta^T\Gamma_{17}\psi &= \zeta_1\psi_7 + \zeta_2\psi_8 - \zeta_3\psi_5 - \zeta_4\psi_6 + \zeta_5\psi_3 + \zeta_6\psi_4 - \zeta_7\psi_1 - \zeta_8\psi_2 \\
\zeta^T\Gamma_{18}\psi &= -\zeta_1\psi_8 + \zeta_2\psi_7 + \zeta_3\psi_6 - \zeta_4\psi_5 + \zeta_5\psi_4 - \zeta_6\psi_3 - \zeta_7\psi_2 + \zeta_8\psi_1 \\
\zeta^T\Gamma_{23}\psi &= \zeta_1\psi_3 + \zeta_2\psi_4 - \zeta_3\psi_1 - \zeta_4\psi_2 + \zeta_5\psi_7 + \zeta_6\psi_8 - \zeta_7\psi_5 - \zeta_8\psi_6 \\
\zeta^T\Gamma_{24}\psi &= -\zeta_1\psi_6 - \zeta_2\psi_5 + \zeta_3\psi_8 + \zeta_4\psi_7 + \zeta_5\psi_2 + \zeta_6\psi_1 - \zeta_7\psi_4 - \zeta_8\psi_3 \\
\zeta^T\Gamma_{25}\psi &= -\zeta_1\psi_5 + \zeta_2\psi_6 + \zeta_3\psi_7 - \zeta_4\psi_8 + \zeta_5\psi_1 - \zeta_6\psi_2 - \zeta_7\psi_3 + \zeta_8\psi_4 \\
\zeta^T\Gamma_{26}\psi &= \zeta_1\psi_4 - \zeta_2\psi_3 + \zeta_3\psi_2 - \zeta_4\psi_1 - \zeta_5\psi_8 + \zeta_6\psi_7 - \zeta_7\psi_6 + \zeta_8\psi_5 \\
\zeta^T\Gamma_{27}\psi &= -\zeta_1\psi_8 + \zeta_2\psi_7 - \zeta_3\psi_6 + \zeta_4\psi_5 - \zeta_5\psi_4 + \zeta_6\psi_3 - \zeta_7\psi_2 + \zeta_8\psi_1 \\
\zeta^T\Gamma_{28}\psi &= -\zeta_1\psi_7 - \zeta_2\psi_8 + \zeta_3\psi_5 - \zeta_4\psi_6 + \zeta_5\psi_3 + \zeta_6\psi_4 + \zeta_7\psi_1 + \zeta_8\psi_2 \\
\zeta^T\Gamma_{34}\psi &= \zeta_1\psi_8 + \zeta_2\psi_7 + \zeta_3\psi_6 + \zeta_4\psi_5 - \zeta_5\psi_4 - \zeta_6\psi_3 - \zeta_7\psi_2 - \zeta_8\psi_1 \\
\zeta^T\Gamma_{35}\psi &= \zeta_1\psi_7 - \zeta_2\psi_8 + \zeta_3\psi_5 - \zeta_4\psi_6 - \zeta_5\psi_3 + \zeta_6\psi_4 - \zeta_7\psi_1 + \zeta_8\psi_2 \\
\zeta^T\Gamma_{36}\psi &= \zeta_1\psi_2 - \zeta_2\psi_1 - \zeta_3\psi_4 + \zeta_4\psi_3 - \zeta_5\psi_6 + \zeta_6\psi_5 + \zeta_7\psi_8 - \zeta_8\psi_7 \\
\zeta^T\Gamma_{37}\psi &= -\zeta_1\psi_6 + \zeta_2\psi_5 + \zeta_3\psi_8 - \zeta_4\psi_7 - \zeta_5\psi_2 + \zeta_6\psi_1 + \zeta_7\psi_4 - \zeta_8\psi_3 \\
\zeta^T\Gamma_{38}\psi &= -\zeta_1\psi_5 - \zeta_2\psi_6 + \zeta_3\psi_7 + \zeta_4\psi_8 + \zeta_5\psi_1 + \zeta_6\psi_2 - \zeta_7\psi_3 - \zeta_8\psi_4 \\
\zeta^T\Gamma_{45}\psi &= \zeta_1\psi_2 - \zeta_2\psi_1 + \zeta_3\psi_4 - \zeta_4\psi_3 + \zeta_5\psi_6 - \zeta_6\psi_5 + \zeta_7\psi_8 - \zeta_8\psi_7 \\
\zeta^T\Gamma_{46}\psi &= -\zeta_1\psi_7 + \zeta_2\psi_8 + \zeta_3\psi_5 - \zeta_4\psi_6 - \zeta_5\psi_3 + \zeta_6\psi_4 + \zeta_7\psi_1 - \zeta_8\psi_2 \\
\zeta^T\Gamma_{47}\psi &= -\zeta_1\psi_3 + \zeta_2\psi_4 + \zeta_3\psi_1 - \zeta_4\psi_2 + \zeta_5\psi_7 - \zeta_6\psi_8 - \zeta_7\psi_5 + \zeta_8\psi_6
\end{aligned}$$

$$
\begin{aligned}
\zeta^T\Gamma_{48}\psi &= -\zeta_1\psi_4 - \zeta_2\psi_3 + \zeta_3\psi_2 + \zeta_4\psi_1 - \zeta_5\psi_8 - \zeta_6\psi_7 + \zeta_7\psi_6 + \zeta_8\psi_5 \\
\zeta^T\Gamma_{56}\psi &= \zeta_1\psi_8 + \zeta_2\psi_7 - \zeta_3\psi_6 - \zeta_4\psi_5 + \zeta_5\psi_4 + \zeta_6\psi_3 - \zeta_7\psi_2 - \zeta_8\psi_1 \\
\zeta^T\Gamma_{57}\psi &= \zeta_1\psi_4 + \zeta_2\psi_3 - \zeta_3\psi_2 - \zeta_4\psi_1 - \zeta_5\psi_8 - \zeta_6\psi_7 + \zeta_7\psi_6 + \zeta_8\psi_5 \\
\zeta^T\Gamma_{58}\psi &= -\zeta_1\psi_3 + \zeta_2\psi_4 + \zeta_3\psi_1 - \zeta_4\psi_2 - \zeta_5\psi_7 + \zeta_6\psi_8 + \zeta_7\psi_5 - \zeta_8\psi_6 \\
\zeta^T\Gamma_{67}\psi &= \zeta_1\psi_5 + \zeta_2\psi_6 + \zeta_3\psi_7 + \zeta_4\psi_8 - \zeta_5\psi_1 - \zeta_6\psi_2 - \zeta_7\psi_3 - \zeta_8\psi_4 \\
\zeta^T\Gamma_{68}\psi &= -\zeta_1\psi_6 + \zeta_2\psi_5 - \zeta_3\psi_8 + \zeta_4\psi_7 - \zeta_5\psi_2 + \zeta_6\psi_1 - \zeta_7\psi_4 + \zeta_8\psi_3 \\
\zeta^T\Gamma_{78}\psi &= -\zeta_1\psi_2 + \zeta_2\psi_1 - \zeta_3\psi_4 + \zeta_4\psi_3 + \zeta_5\psi_6 - \zeta_6\psi_5 + \zeta_7\psi_8 - \zeta_8\psi_7 \,.
\end{aligned}
$$

So we get, for example,

$$
(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{12}\psi) = \left(\sum_{A=1}^{8} \bar{\psi}_A\zeta_A\right) \cdot \sum_{a=1}^{4}(-1)^a \left(\zeta_{2a-1}\psi_{2a} - \zeta_{2a}\psi_{2a-1}\right). \tag{3.2}
$$

If we write $\psi = \chi + i\eta$, then the real and imaginary parts of (3.2) read

$$
\begin{aligned}
\mathrm{Re}\left[(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{12}\psi)\right] &= \left(\sum_{A=1}^{8} \chi_A\zeta_A\right) \cdot \sum_{a=1}^{4}(-1)^a \left(\zeta_{2a-1}\chi_{2a} - \zeta_{2a}\chi_{2a-1}\right) + \\
&\quad + \left(\sum_{A=1}^{8} \eta_A\zeta_A\right) \cdot \sum_{a=1}^{4}(-1)^a \left(\zeta_{2a-1}\eta_{2a} - \zeta_{2a}\eta_{2a-1}\right), \qquad (3.3) \\
\mathrm{Im}\left[(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{12}\psi)\right] &= \left(\sum_{A=1}^{8} \chi_A\zeta_A\right) \cdot \sum_{a=1}^{4}(-1)^a \left(\zeta_{2a-1}\eta_{2a} - \zeta_{2a}\eta_{2a-1}\right) - \\
&\quad - \left(\sum_{A=1}^{8} \eta_A\zeta_A\right) \cdot \sum_{a=1}^{4}(-1)^a \left(\zeta_{2a-1}\chi_{2a} - \zeta_{2a}\chi_{2a-1}\right). \qquad (3.4)
\end{aligned}
$$

Other (μ,ν)-components can be written down similarly.

The above explicit result shows that for generic spinors ψ, both the real part and the imaginary part of $\mathcal{Q}_{\mu\nu}(\psi,\bar{\psi})$ are indeed not zero. This is different from the 4-dimensional Seiberg-Witten theory, where the quadratic form $\mathcal{Q}_{\mu\nu}(\psi,\bar{\psi}) = \bar{\psi}^T\Gamma_{\mu\nu}\psi$ is essentially purely imaginary, as we can choose the Spin(4) Lie algebra generators $\Gamma_{\mu\nu}$ to be anti-hermitean. The difference stems from the fact that in 4-dimensional theory the quadratic takes the "diagonal form" $\bar{\psi}^T\Gamma_{\mu\nu}\psi \in \Lambda^2_+$ while in eight dimensions, such a diagonal form does not belong to Λ^2_+ (though it is still purely imaginary). In order to define a reasonable $\mathcal{Q} \in \Lambda^2_+$ in 8 dimensions, we have to decompose the spinor $\psi \in S^+ \cong \mathbf{1}\oplus\mathbf{7}$ into two parts $\psi = \psi_{\mathbf{1}}+\psi_{\mathbf{7}}$, one of which, $\psi_{\mathbf{1}} \equiv (\zeta^T\psi)\zeta$, is in $\mathbf{1}$, i.e. the trivial module of Spin(7), and the other of which, $\psi_{\mathbf{7}}$, belongs to $\mathbf{7}$, namely the seven-dimensional irreducible module of Spin(7). Since this decomposition is orthogonal and since $\Gamma_{\mu\nu}\zeta \in \mathbf{7}$, we have $\bar{\psi}^T\zeta = \bar{\psi}^T_{\mathbf{1}}\zeta$ and $\zeta^T\Gamma_{\mu\nu}\psi = \zeta^T\Gamma_{\mu\nu}\psi_{\mathbf{7}}$. Thus, the quadratic form $\mathcal{Q}_{\mu\nu} = (\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi) = (\bar{\psi}^T_{\mathbf{1}}\zeta)(\zeta^T\Gamma_{\mu\nu}\psi_{\mathbf{7}})$ we have just constructed is really an "off-diagonal" product between the independent degrees of freedom $\psi_{\mathbf{1}}$ and $\psi_{\mathbf{7}}$. Such a product cannot be automatically real or purely imaginary. This explains why in the first equation of (3.1), we have splitted the quadratic form into its real and imaginary parts, and introduced two real coefficients a and b.

There is a more compact way to write down the real and imaginary parts of $\mathcal{Q}$:

$$
\begin{aligned}
\mathrm{Re}\left[(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi)\right] &= \frac{1}{2\cdot 4!}\Omega^{\lambda\rho\sigma}{}_{[\mu}(\psi^\dagger\Gamma_{\nu]\lambda\rho\sigma}\psi)\,, \\
i\,\mathrm{Im}\left[(\bar{\psi}^T\zeta)(\zeta^T\Gamma_{\mu\nu}\psi)\right] &= \frac{1}{8}(\psi^\dagger\Gamma_{\mu\nu}\psi) - \frac{1}{16}\Omega_{\mu\nu\alpha\beta}(\psi^\dagger\Gamma^{\alpha\beta}\psi) \\
&\equiv \frac{1}{2}(P^+)_{\mu\nu}{}^{\alpha\beta}(\psi^\dagger\Gamma_{\alpha\beta}\psi)
\end{aligned}
\tag{3.5}
$$

where $P^+ : \Lambda^2 \to \Lambda^2_+$ is the orthorgonal projection [7] of two-forms onto Λ^2_+. Note that the imaginary part of $\mathcal{Q}$ resembles the term $\psi^\dagger\Gamma_{\mu\nu}\psi$ in the Seiberg-Witten theory, but in eight dimensions there are additional ingredients in the construction of a general quadratic form valued in Λ^2_+: we have terms involving $\Omega^{\lambda\rho\sigma}{}_{[\mu}\Gamma_{\nu]\lambda\rho\sigma}$. Such terms are forbidden in 4 dimensions since there $\Omega^{\lambda\rho\sigma}{}_\mu \propto \epsilon^{\lambda\rho\sigma}{}_\mu$, $\Gamma_{\nu\lambda\rho\sigma} \propto \epsilon_{\nu\lambda\rho\sigma}$ and $\epsilon^{\lambda\rho\sigma}{}_{[\mu}\epsilon_{\nu]\lambda\rho\sigma} = 0$.

Let us discuss another difference between the 8-dimensional and 4-dimensional theories. Writing down the equations in such theories requires to fix certain geometrical data on the underlying manifold. For example, in order to construct the anti self-dual part F^+ of the curvature tensor in the 4-dimensional theory, one has to pick up a Hodge star operator, whose definition depends on the conforml structure of the manifold. Thus, the geometrical data — a conformal structure of the 4-manifold — enters natually in the first Seiberg-Witten equation $F^+_{\mu\nu} \sim \mathcal{Q}_{\mu\nu}$. Such geometrical data also enters in the the second Seiberg-Witten equation, i.e. the massless Dirac equation in 4 dimensions, as that equation is conformally invariant and it also depends on the choice of a conformal structure. In the 8-dimensional theory, the construction of the first equation involves another data, Ω, which is the Spin(7)-invariant 4-form calibrating the underlying geometry. This can be expected, since as long as the self-duality structures are concerned Ω will play a role similar to the Hodge star operator in 4 dimensions. What makes the 8-dimensional theory different from that in 4 dimensions is that the geometrical data Ω does not enter in the Dirac equation. Thus, it should not be very suprising when we find that the functional formalism of (3.1) in general does not allow the delicate cancellations as in the 4-dimensional theory. In particular, we do not know at present how to handle the uncancelled terms involving F^-, arsing from the functional $||\Gamma_\mu D^\mu_A\psi||^2$ of the Dirac equation. One possible resolusion is to modify the second equation in (3.1) so that it depends on the form Ω (through the Spin(7)-invariant spinor ζ).

Manifolds with $\mathrm{SU}(4)$ **holonomy.** Now we try to formulate an eight dimensional analog of the Seiberg-Witten equations on manifolds with the $\mathrm{SU}(4)$ holonomy group. The starting point will be similar to that in the Spin(7) case: one wishes to replace the nonabelian instanton equation $F^{(0,2)}_{+\bar i\bar j}=0$ by an abelian, monopole-like equation $F^{(0,2)}_{+\bar i\bar j}=\mathcal{Q}_{\bar i\bar j}(\psi,\bar\psi)$, where $\psi\in S^+_c\otimes\mathcal{L}$ is a spinor field twisted by some complex line bundle $\mathcal{L}$, and $\mathcal{Q}_{\bar i\bar j}(\psi,\bar\psi)$ denotes a certain quadratic form valued in $\Lambda^{0,2}_+$. Our first task is thus to find out such a quadratic form.

The condition for a $(0,2)$-form $\beta_{\bar i\bar j}$ to be valued in $\Lambda^{0,2}_+$ is that it obeys the eigenvalue equation $*_\Omega\beta=-4\beta$. So according to (2.28), $\beta\in\Lambda^{0,2}_+$ is characterized by the equations

$$\bar\Omega_{\bar i\bar j\bar k\bar l}\bar\beta^{\bar k\bar l}=-8\beta_{\bar i\bar j}\quad\Longleftrightarrow\quad\Omega_{ijkl}\beta^{kl}=-8\bar\beta_{ij}\,. \tag{3.6}$$

Our key observation here is that the spinor $\gamma^{ij}\bar\zeta$ satisfies an equation with the same structure as the second one in (3.6). To see this, multiplying (2.26) by $\zeta^T\gamma_{ijkl}$ from the left, we find

$$\Omega_{ijkl}\bar\zeta=\frac{1}{8}\zeta^T(\gamma_{ijkl}+g^{m\bar n}\gamma_{ijkl}\gamma_{m\bar n})+\frac{1}{32}\zeta^T g^{m\bar s}g^{n\bar r}\gamma_{ijkl}\gamma_{mn\bar r\bar s}\,.$$

Notice that $\gamma_{i_1\cdots i_p}=\gamma_{i_1}\cdots\gamma_{i_p}$ and $\gamma_{i_1\cdots i_p\bar j_1\cdots\bar j_q}=0$ for $p>4$. So one can use eq. (2.21) repeatedly to compute $g^{m\bar n}\gamma_{ijkl}\gamma_{m\bar n}$ as well as $g^{m\bar s}g^{n\bar r}\gamma_{ijkl}\gamma_{mn\bar r\bar s}$, and the result simply reads

$$g^{m\bar n}\gamma_{ijkl}\gamma_{m\bar n}=4\gamma_{ijkl}\,,\qquad g^{m\bar s}g^{n\bar r}\gamma_{ijkl}\gamma_{mn\bar r\bar s}=12\gamma_{ijkl}\,.$$

Consequently, we have [10]

$$\Omega_{ijkl}\bar\zeta=\zeta^T\gamma_{ijkl}=\gamma_{ijkl}\zeta \tag{3.7}$$

(where the last identity comes from the symmetric property of the matrix γ_{ijkl}). Now with the help of (3.7) and (2.21), we can do some further computations:

$$\begin{aligned}\Omega_{ijkl}\gamma^{kl}\bar\zeta&=\gamma^{kl}(\Omega_{ijkl}\bar\zeta)=\gamma^{kl}\gamma_{ijkl}\zeta=g^{k\bar m}g^{l\bar n}\gamma_{\bar m\bar n}\gamma_{ijkl}\zeta\\&=g^{k\bar m}g^{l\bar n}\gamma_{\bar m}\gamma_{ijkl\bar n}\zeta+g^{k\bar m}\gamma_{\bar m}\gamma_{ijk}\zeta\\&=g^{k\bar m}g^{l\bar n}\gamma_{ijkl\bar m\bar n}\zeta-2g^{k\bar m}\gamma_{ijk\bar m}\zeta-2\gamma_{ij}\zeta=-8\gamma_{ij}\zeta\,.\end{aligned}$$

So finally we arrive at

$$\Omega_{ijkl}\gamma^{kl}\bar\zeta=-8\gamma_{ij}\zeta\,, \tag{3.8}$$

which shows that $\gamma^{ij}\bar\zeta$ has the same tensor properties as an anti self-dual two-form $\beta^{ij}\in\Lambda^{0,2}_+$.

Thus, given any spinor $\psi\in S^+_c\otimes\mathcal{L}$, one can use the isomorphism (2.22) to construct a form $\beta^{ij}=-\psi^T\gamma^{ij}\bar\zeta=\zeta^\dagger\gamma^{ij}\psi\in(S^+_c\otimes\mathcal{L})\otimes\mathbb{C}\cong\Lambda^{0,\mathrm{even}}\otimes\mathcal{L}$. Naively, the identity (3.8) indicates that such a form should obey the anti self-duality equation (3.6), and thus it would belong to the subspace $\Lambda^{0,2}_+\otimes\mathcal{L}$:

$$\beta^{ij}\equiv\zeta^\dagger\gamma^{ij}\psi\in\Lambda^{0,2}_+\otimes\mathcal{L}\,. \tag{3.9}$$

To construct a quadratic form $\mathcal{Q}_{\bar i\bar j}(\psi,\bar\psi)\in\Lambda^{0,2}_+$, one still needs another form α, which should be anti-linear in ψ and valued in $\Lambda^{0,0}\otimes\mathcal{L}^{-1}$, so that the factor $\mathcal{L}$ could be cancelled when forming the product $\alpha\beta_{\bar{i}\bar{j}}$. The simplest choice of such a form would be

$$\alpha \equiv \zeta^T \bar{\psi} = \psi^\dagger \zeta \in \Lambda^{0,0} \otimes \mathcal{L}^{-1} \,. \tag{3.10}$$

So at first sight we expect that the quadratic form we are seeking should look like:[1]

$$\mathcal{Q}_{\bar{i}\bar{j}}(\psi, \bar{\psi}) \sim \alpha\beta_{\bar{i}\bar{j}} = (\psi^\dagger \zeta)(\zeta^\dagger \gamma_{\bar{i}\bar{j}} \psi) \,. \tag{3.11}$$

However, there is a subtlety in the above construction, which appears only in the complex case. In our definition of self-duality, the star operator $*_\Omega$ given in (2.28) is conjugate-linear rather than linear. Thus, even if $\beta_{\bar{i}\bar{j}}$ is anti self-dual, namely it obeys the condition (3.6), the quantity $\alpha\beta_{\bar{i}\bar{j}}$ needs not to be such a form for *complex* $\alpha \in \Lambda^{0,0} \otimes \mathcal{L}^{-1}$. We cannot simply take α to be real as $\mathcal{L}$ should be a nontrivial complex line bundle. Moreover, since ψ is also a complex spinor, the $(0,2)$-form $\beta^{ij} = -\psi^T \gamma^{ij} \bar{\zeta}$ defined by (3.9) is not really valued in $\Lambda_+^{0,2}$, even though $\gamma^{ij}\bar{\zeta}$ behaves as an anti self-dual tensor. To solve this problem, let us introduce a pair α, α' of $(0,0)$-forms as well as a pair $\beta_{\bar{i}\bar{j}}$, $\beta'_{\bar{i}\bar{j}}$ of $(0,2)$-forms, specified as

$$\begin{aligned} \alpha &= \psi^\dagger \zeta \in \Lambda^{0,0} \otimes \mathcal{L}^{-1} \,, & \alpha' &= \bar{\alpha} = \zeta^\dagger \psi \in \Lambda^{0,0} \otimes \mathcal{L} \\ \beta^{ij} &= \zeta^\dagger \gamma^{ij} \psi \in \Lambda^{0,2} \otimes \mathcal{L}, & \beta'^{ij} &= \zeta^\dagger \gamma^{ij} \bar{\psi} \in \Lambda^{0,2} \otimes \mathcal{L}^{-1} \,, \end{aligned} \tag{3.12}$$

and construct the product

$$\mathcal{Q}_{\bar{i}\bar{j}}(\psi, \bar{\psi}) \equiv c\alpha\beta_{\bar{i}\bar{j}} + \bar{c}\alpha'\beta'_{\bar{i}\bar{j}} = c(\psi^\dagger\zeta)(\zeta^\dagger\gamma_{\bar{i}\bar{j}}\psi) + \bar{c}(\psi^T\bar{\zeta})(\zeta^\dagger\gamma_{\bar{i}\bar{j}}\bar{\psi}) \,, \tag{3.13}$$

with $c \in \mathbb{C}$ being an arbitrarily fixed complex number (similar to the real numbers a, b in the Spin(7) case). One then uses (3.8) to derive

$$\begin{aligned} \Omega_{ijkl}\mathcal{Q}^{kl} &= -c(\psi^\dagger\zeta)(\psi^T\Omega_{ijkl}\gamma^{kl}\bar{\zeta}) - \bar{c}(\psi^T\bar{\zeta})(\psi^\dagger\Omega_{ijkl}\gamma^{kl}\bar{\zeta}) \\ &= 8c(\psi^\dagger\zeta)(\psi^T\gamma_{ij}\zeta) + 8\bar{c}(\psi^T\bar{\zeta})(\psi^\dagger\gamma_{ij}\zeta) = -8\bar{\mathcal{Q}}_{ij} \,, \end{aligned}$$

which indicates that $\mathcal{Q}_{\bar{i}\bar{j}}$ is now in $\Lambda_+^{0,2}$.

Having constructed a quadratic form $\mathcal{Q}_{\bar{i}\bar{j}}$ with the right properties, we can immediately write down the first equation analogous to Seiberg and Witten's:

$$F_{+\bar{i}\bar{j}}^{(0,2)} = c\alpha\beta_{\bar{i}\bar{j}} + \bar{c}\alpha'\beta'_{\bar{i}\bar{j}} = c(\psi^\dagger\zeta)(\zeta^\dagger\gamma_{\bar{i}\bar{j}}\psi) + \bar{c}(\psi^\dagger\gamma_{\bar{i}\bar{j}}\bar{\zeta})(\zeta^\dagger\psi) \,. \tag{3.14}$$

One may also write down a similar equation for $F_{+ij}^{(2,0)}$ by taking the complex conjugate of (3.14). It should be pointed out, however, that at this stage we have not yet

[1] (3.11) is very similar to the quadratic form $\mathcal{Q}_{\mu\nu}$ constructed in the real case. One may decompose ψ into $\psi = \psi_\parallel + \psi_\perp$, where $\psi_\parallel$ is valued in the SU(4) invariant subspace of S_c^+ spanned by ζ and $\bar{\zeta}$, and $\psi_\perp$ lives in the subspace orthogonal to it. Using the facts that $\zeta^T\zeta = 0$, $\zeta^\dagger\gamma_{\bar{i}\bar{j}}\zeta = \zeta^\dagger\gamma_{\bar{i}\bar{j}}\bar{\zeta} = 0$, we find that (3.11) can be represented as an "off-diagonal" product of the two independent degrees of freedom $\psi_\parallel$ and $\psi_\perp$, namely $\mathcal{Q}_{\bar{i}\bar{j}} = (\psi_\parallel^\dagger\zeta)(\zeta^\dagger\gamma_{\bar{i}\bar{j}}\psi_\perp)$. This resembles the Spin(7) case, where the quadratic form is also an off-diagonal product of two independent degrees of freedom.

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established another kind of equation (something like $F_\omega^{(1,1)} \sim -\frac{1}{2}\omega(||\alpha||^2 - ||\beta||^2)$ as in the 4-dimensional theory), which governs the $(1,1)$-component of F in the direction along the Käller form $\omega_{i\bar{j}} = ig_{i\bar{j}}$. To obtain such an equation, one should use a new star operator $*_\Theta$ with $\Theta \sim \mathrm{Re}(\Omega) + \omega^2$ to define self-duality [8].

Next we consider the Dirac equation $D_A\psi = 0$. With the isomorphism $(S_c^+, S_c^-)\otimes \mathbb{C} \cong (\Lambda^{0,\mathrm{even}}, \Lambda^{0,\mathrm{odd}})$, the twisted Dirac operator $D_A : S_c^+ \otimes \mathcal{L} \to S_c^- \otimes \mathcal{L}$ becomes $D_A = \bar{\partial}_A + \bar{\partial}_A^* : \Lambda^{0,\mathrm{even}} \otimes \mathcal{L} \to \Lambda^{0,\mathrm{odd}} \otimes \mathcal{L}$. Let us restrict this operator to $(\Lambda^{0,0} \oplus \Lambda^{0,2}) \otimes \mathcal{L}$. Under this restriction, the spinor $\psi \in \Lambda^{0,\mathrm{even}} \otimes \mathcal{L}$ has the components $\alpha' = \bar{\alpha} \in \Lambda^{0,0} \otimes \mathcal{L}$ and $\beta \in \Lambda^{0,2} \otimes \mathcal{L}$, and the Dirac equation is reduced simply to

$$\bar{\partial}_A \bar{\alpha} + \bar{\partial}_A^* \beta = 0\,. \tag{3.15}$$

This constitutes the second equation in our theory.

Although eq. (3.14)-(3.15) resemble the four-dimensional Seiberg-Witten equations on Kähler manifolds [12], it should be pointed out that here the Dirac equation (3.15) in general does not allow a simple decomposition into $\bar{\partial}_A \bar{\alpha} = \bar{\partial}_A^* \beta = 0$. This makes a computation of the relevant invariants quite difficult. This difficulty is related to a problem appeared in the Spin(7) case, where we mentioned that there is an uncancelled term involving F^- in the functional formalism.

4. S-duality in abelian gauge theory

In this section we turn to the abelian gauge theory in eight dimensions. For simplicity, we will consider only the case when X has the holonomy group Spin(7). Classically we have a U(1) gauge field A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, together with the following action functional

$$\begin{aligned} S[A] &= \frac{1}{2g^2}\int_X F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{8\pi^2}\int_X \Omega \wedge F \wedge F \\ &= \frac{1}{2g^2}\int_X F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{32\pi^2}\int_X \Omega_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\,. \end{aligned} \tag{4.1}$$

Given such data, the partition function $Z(g,\theta)$ can be formally defined as the euclidean path-integral

$$Z(g,\theta) = \int [dA]\, e^{-S[A]}\,. \tag{4.2}$$

Let us analyze the partition function in some detail. Usually, it is convenient to change the integration variables $A \to F$. The routine is quite standard: just as in four dimensions, F is not an independent variable, and it must be subject to the Bianchi identity $dF = 0$. If we write $dF = \frac{1}{2}\partial_\lambda F_{\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu$, then one easily deduces from the closeness and self-duality of Ω that

$$dF \wedge \Omega \wedge dx^\nu = \frac{1}{2}\partial_\mu[(*_\Omega F)^{\mu\nu}]\, d(\mathrm{vol})\,.$$

This implies that the Bianchi identity $dF = 0$ can be replaced by a constraint

$$\partial_\nu[(*_\Omega F)^{\mu\nu}] = 0 \tag{4.3}$$

on the field strength. Consequently, the partition function (4.2) has a path-integral representation over F, with the delta function $\delta(\partial_\nu[(*_\Omega F)^{\mu\nu}])$ inserted. Such a delta function can be written as another path-integral over some auxiliary field A^D_μ. Thus, one may write

$$\begin{aligned} Z(g,\theta) &= \int [dF][dA^D]\, e^{-S+i\int_X A^D_\mu \partial_\nu (*_\Omega F)^{\mu\nu}} \\ &= \int [dF][dA^D]\, e^{-S+\frac{i}{2}\int_X F^D_{\mu\nu}(*_\Omega F)^{\mu\nu}}\,, \end{aligned} \tag{4.4}$$

where $F^D_{\mu\nu} = \partial_\mu A^D_\nu - \partial_\nu A^D_\mu$ is the field strength of A^D_μ. If we first integrate out the auxiliary field A^D, then the resulting expressonis is nothing but $\int [dF]\delta(\partial_\nu[(*_\Omega F)^{\mu\nu}])e^{-S}$, which is equivalent to the original partition function (4.2).

Alternatively, one can integrate out F in (4.4) first, leaving an effective action for the auxiliary field, which governs the dynamics of the collective variable A^D_μ. To achieve this, let us decompose $F = F^+ + F^-$ into two independent components F^+, F^- and write $[dF] = [dF^+][dF^-]$. In terms of these components, we have

$$\begin{aligned} S &= \left(\frac{1}{2g^2} - i\frac{3\theta}{16\pi^2}\right)\int_X F^+_{\mu\nu}F^{+\mu\nu} + \left(\frac{1}{2g^2} + i\frac{\theta}{16\pi^2}\right)\int_X F^-_{\mu\nu}F^{-\mu\nu} \\ &\equiv \frac{3}{4e^2_+}\int_X F^+_{\mu\nu}F^{+\mu\nu} + \frac{1}{4e^2_-}\int_X F^-_{\mu\nu}F^{-\mu\nu}\,, \end{aligned} \tag{4.5}$$

$$\frac{i}{2}\int_X F^D_{\mu\nu}(*_\Omega F)^{\mu\nu} = -\frac{3i}{2}\int_X F^{D+}_{\mu\nu}F^{+\mu\nu} + \frac{i}{2}\int_X F^{D-}_{\mu\nu}F^{-\mu\nu}\,. \tag{4.6}$$

So substituting (4.5)-(4.6) into (4.4) yields a product of two gaussian integrals over $F^\pm$. An explicit evaluation of these integrals gives

$$\begin{aligned} Z(g,\theta) &= \int [dA^D]\, e^{-\widetilde{S}[A^D]}\,, \\ \widetilde{S}[A^D] &= \frac{3e^2_+}{4}\int_X F^{D+}_{\mu\nu}F^{D+\mu\nu} + \frac{e^2_-}{4}\int_X F^{D-}_{\mu\nu}F^{D-\mu\nu}\,. \end{aligned} \tag{4.7}$$

We thus obtain a dual description of the original theory using the collective field A^D, in which the coupling constants get transformed:

$$e^2_\pm \to \widetilde{e}^2_\pm = \frac{1}{e^2_\pm}\,. \tag{4.8}$$

Eq. (4.8) characterizes a generalized S-duality in eight dimensions.

The above discussion is somewhat rough and we ignored several subtleties arising from regularization. In four dimensions, a more careful study [16] shows that the partition function transforms as a modular form, and this provides a precise test of the S-duality. When entering in eight dimensions, however, one sees from (4.5) that the action does not takes the form $S \propto i \int (\tau (F^+)^2 - \bar{\tau}(F^-)^2)$, so the partition function $Z(g, \theta)$ will not be parametrized neatly by a single complex coupling τ along with its conjugate $\bar{\tau}$; more naturally, $Z(g, \theta)$ should be parametrized by (e^+, e^-), and $e^{\pm}$ are not complex conjugate to each other. It seems rather difficult to write down a simple modular form expression for the partition function of the eight-dimensional theory. Without such a modular form our understanding of the generalized S-duality is quite incomplete.

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A. γ-matrices and Clifford calculus

In the text we used Cl(8) to denote the 8-dimensional Clifford algebra. This algebra has a real, irreducible representation $\rho : \mathrm{Cl}(8) \to \mathrm{End}(S)$. According to the standard argument, $\rho(\mathrm{Cl}(8))$ constitutes the algebra $\mathbb{R}(16)$ of 16×16 real matrices, acting on the 16-dimensional vector space $S \cong \mathbb{R}^{16}$. The following isomorphism between Cl(8) and the wedge algebra $\wedge^* \mathbb{R}^8$ is quite evident:

$$\mathrm{Cl}(8) \cong \wedge^* \, \mathbb{R}^8 \, . \tag{A.1}$$

In particular, if we introduce a set of orthogonal generators of Cl(8), $e_\mu \in \wedge^1 \mathbb{R}^8$ $(1 \le \mu \le 8)$, with the rule of Clifford multiplications

$$e_\mu \cdot e_\nu + e_\nu \cdot e_\mu = -2 \, \langle e_\mu, e_\nu \rangle \equiv -2 g_{\mu\nu} \, , \tag{A.2}$$

then the p-"form" $e_{\mu_1} \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_p} \in \wedge^p \mathbb{R}^8$ canonically has the representation

$$e_{\mu_1} \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_p} \longleftrightarrow \Gamma_{\mu_1 \mu_2 \cdots \mu_p} \equiv \Gamma_{[\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_p]} \, , \tag{A.3}$$

where

$$\Gamma_\mu = \rho(e_\mu) \in \mathbb{R}(16) \tag{A.4}$$

are known as "γ-matrices", and the square bracket indicates anti-symmetrization of the indices, with a prefactor $1/p!$. We shall use the notation $\Gamma_{\mu_1 \cdots \mu_p} = I_{16 \times 16}$ for $p = 0$.

The Clifford multiplication "$\cdot$" between $u = \sum_\mu C^\mu e_\mu \in \wedge^1 \mathbb{R}^8 \subset \mathrm{Cl}(8)$ and any element $w \in \mathrm{Cl}(8) \cong \wedge^* \mathbb{R}^8$ can be identified with an operation on the wedge algebra:

$$u \cdot w \longleftrightarrow u \wedge w - i_u(w) \, , \tag{A.5}$$

where the interior product $i_u(w)$ is defined by the linear map $i_u : \wedge^p\mathbb{R}^8 \to \wedge^{p-1}\mathbb{R}^8$, via

$$i_u(u_1 \wedge u_2 \wedge \cdots \wedge u_p) = \sum_{i=1}^{p}(-1)^{i+1}\langle u_i, u\rangle u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_p\,. \tag{A.6}$$

Applying this to the matrix representation ρ, (A.5) becomes an identity between γ-matrices

$$\Gamma_\mu\Gamma_{\nu_1\nu_2\cdots\nu_p} = \Gamma_{\mu\nu_1\nu_2\cdots\nu_p} - g_{\mu\nu_1}\Gamma_{\nu_2\nu_3\cdots\nu_p} + g_{\mu\nu_2}\Gamma_{\nu_1\nu_3\cdots\nu_p} + \cdots + (-1)^p g_{\mu\nu_p}\Gamma_{\nu_1\cdots\nu_{p-1}}\,. \tag{A.7}$$

Sometimes we need to fix a particular basis and construct the γ-matrices Γ_μ explicitly. Our convention of choosing such matrices is as follows. Since $\mathbb{R}(16) \cong \mathbb{R}(2)\otimes\mathbb{R}(2)\otimes\mathbb{R}(2)\otimes\mathbb{R}(2)$, Γ_μ can be expressed by a 4-fold tensor product of some basis in $\mathbb{R}(2)$. Thus, we take a basis of $\mathbb{R}(2)$ to be

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.8}$$

It is easy to checks that the 16×16 matrices

$$\begin{aligned} \Gamma_1 &= \epsilon\otimes\epsilon\otimes\epsilon\otimes\sigma_1, & \Gamma_2 &= I\otimes\sigma_1\otimes\epsilon\otimes\sigma_1, \\ \Gamma_3 &= I\otimes\sigma_3\otimes\epsilon\otimes\sigma_1, & \Gamma_4 &= \sigma_1\otimes\epsilon\otimes I\otimes\sigma_1, \\ \Gamma_5 &= \sigma_3\otimes\epsilon\otimes I\otimes\sigma_1, & \Gamma_6 &= \epsilon\otimes I\otimes\sigma_1\otimes\sigma_1, \\ \Gamma_7 &= \epsilon\otimes I\otimes\sigma_3\otimes\sigma_1, & \Gamma_8 &= I\otimes I\otimes I\otimes\epsilon \end{aligned} \tag{A.9}$$

obey the relations $\{\Gamma_\mu, \Gamma_\nu\} = -2\delta_{\mu\nu}$.

In our convention (A.9), the matrices Γ_μ are all anti-symmetric. More generally we have

$$(\Gamma_{\mu_1\mu_2\cdots\mu_p})^T = (-1)^p\,\Gamma_{\mu_p\cdots\mu_2\mu_1} = (-1)^{\frac{p(p+1)}{2}}\Gamma_{\mu_1\mu_2\cdots\mu_p}\,. \tag{A.10}$$

Thus $\Gamma_{\mu_1\mu_2\cdots\mu_p}$ is anti-symmetric when $p \equiv 1,2 \pmod 4$ and symmetric when $p \equiv 3,4 \pmod 4$. Consequently, the matrix representation of the "volume element" $\omega \equiv e_1\wedge e_2\wedge\cdots\wedge e_8 \in \mathrm{Cl}(8)$, namely

$$\rho(\omega) = \Gamma_1\Gamma_2\cdots\Gamma_8 \equiv \Gamma_9\,, \tag{A.11}$$

has the symmetric property $(\Gamma_9)^T = \Gamma_9$. In fact Γ_9 is diagonal in our basis:

$$\Gamma_9 = I\otimes I\otimes I\otimes\sigma_3 = \begin{pmatrix} I_{8\times8} & 0 \\ 0 & -I_{8\times8} \end{pmatrix}. \tag{A.12}$$

Eq. (A.12) shows that the irreducible Cl(8) module $S \cong \mathbb{R}^{16}$ has a decomposition $S = S^+ \oplus S^-$ into the ±1 eigenspaces $S^\pm$ of Γ_9. Of course neither of these eight-dimensional eigenspaces are invariant under the action of Cl(8). To be a little more explicit, notice that $\Gamma_9\Gamma_{\mu_1\cdots\mu_p} = (-1)^p\Gamma_{\mu_1\cdots\mu_p}\Gamma_9$, we have

$$\Gamma_{\mu_1\cdots\mu_p} \sim \begin{cases} \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}, & p = \text{even} \\ \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}, & p = \text{odd} \end{cases} \quad \text{on} \quad S = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \tag{A.13}$$

so for odd p the matrix $\Gamma_{\mu_1\cdots\mu_p}$ swaps $S^\pm$. Nevertheless, if one considers a subalgebra of Cl(8) spanned by some even elements $a = e_{\mu_1} \wedge \cdots \wedge e_{\mu_{2k}}$, then the matrix representation $\rho(a) = \Gamma_{\mu_1\cdots\mu_{2k}}$ will keep both the subspaces $S^\pm \subset S$ invariant.

At this point we consider a linear space $\wedge^2\mathbb{R}^8$ spanned by elements of the form $L_{\mu\nu} = \frac{1}{2}e_\mu \wedge e_\nu = \frac{1}{4}(e_\mu e_\nu - e_\nu e_\mu)$. This spcae forms a Lie algebra under the bracket

$$[L_{\mu\nu},\, L_{\alpha\beta}] \equiv L_{\mu\nu} \cdot L_{\alpha\beta} - L_{\alpha\beta} \cdot L_{\mu\nu}\,, \tag{A.14}$$

where "$\cdot$" again stands for the Clifford multiplication. In fact, a simple computation shows that

$$[L_{\mu\nu},\, L_{\alpha\beta}] = g_{\mu\alpha}L_{\nu\beta} + g_{\nu\beta}L_{\mu\alpha} - g_{\nu\alpha}L_{\mu\beta} - g_{\mu\beta}L_{\nu\alpha}\,, \tag{A.15}$$

so $\{L_{\mu\nu}\}$ generates the Lie algebra of Spin(8). According to the previous discussion, $S^\pm$ can be considered as Spin(8)-modules, and actually they are two inequivenlent irreducible modules of Spin(8). The representation $\rho(L_{\mu\nu}) = \frac{1}{2}\Gamma_{\mu\nu}$ of (the Lie algebra of) Spin(8) then decomposes into two irreducible ones: $\rho = \rho^+ \oplus \rho^-$. ρ^+ is the spin representation with positive chirality and ρ^- the spin representation with negative chirality. That $\rho^\pm$ are inequivalent stems from the "central element" $\Gamma_9 = \rho(\omega) = \pm 1$ having different values on $S^\pm$.

So far we have only constructed two irreducible spin representations of Spin(8), $\rho^\pm$, acting on $S^+ = \mathbf{8}_s$ and $S^- = \mathbf{8}_c$, respectively. There is another inequivalent eight-dimentional irreducible representation of Spin(8), the so-called "vector representation" ρ_v, which will act on the vector space $\mathbf{8}_v \equiv Span\{e_\mu\} \cong \wedge^1\mathbb{R}^8$ adjointly:

$$\rho_v(L_{\mu\nu})(e_\alpha) \equiv [L_{\mu\nu}, e_\alpha]\,. \tag{A.16}$$

The matrix elements of $\rho_v(L_{\mu\nu})$ are determined by the following commutative relations:

$$[L_{\mu\nu}, e_\alpha] = g_{\mu\alpha}e_\nu - g_{\nu\alpha}e_\mu\,. \tag{A.17}$$

It follows that the image of Spin(8) under ρ_v is isomorphic to SO(8). The existence of the three inequivalent irreducible 8-dimensional modules $\mathbf{8}_s$, $\mathbf{8}_c$ and $\mathbf{8}_v$ is often summarized as the triality of the Spin(8)-representations.

As a natural extension of the vector representation ρ_v, it is possible to construct tensor representations of Spin(8) on $\wedge^p\mathbb{R}^8$. One verifies by induction that (A.17) is extended to

$$[L_{\mu\nu}, e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p}] = M_{\alpha_1\cdots\alpha_p}^{\beta_1\cdots\beta_p}(L_{\mu\nu})e_{\beta_1} \wedge \cdots \wedge e_{\beta_p}\,, \tag{A.18}$$

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with some adjoint matrices $M(L_{\mu\nu})$. This therefore defines a tensor representation $\wedge^p\rho_v$ of Spin(8). Alternatively, $\wedge^p\rho_v$ may also be obtained by considering tensor product of the spin representations $\rho^\pm$.

To see this, we first need to establish an isomorphism between the spaces $(S^+\oplus S^-)\otimes(S^+\oplus S^-)$ and $\wedge^*\mathbb{R}^8$. Note that both spaces have the same dimensions: $16\times 16 = 2^8$. By choosing an orthogonal basis $\{v_A\}_{1\leq A\leq 16}$ of $S = S^+\oplus S^-$, we associate each $v_A\otimes v_B \in S\otimes S$ to an element of $\wedge^*\mathbb{R}^8$ as follows:

$$v_A\otimes v_B \longleftrightarrow \bigoplus_{p=0}^{8}\langle v_A, \Gamma^{\mu_1\cdots\mu_p}v_B\rangle\, e_{\mu_1}\wedge\cdots\wedge e_{\mu_p}\,. \tag{A.19}$$

This correspondence is $1:1$ since if $v_A\otimes v_B$, $v_C\otimes v_D$ are associated to the same element of $\wedge^*\mathbb{R}^8$, then we must have $\langle v_A,\Gamma^{\mu_1\cdots\mu_p}v_B\rangle \equiv \langle v_C,\Gamma^{\mu_1\cdots\mu_p}v_D\rangle$ for all p and, by irreducibility of the Clifford group[2] acting on S, the two matrix elements must be orthogonal and never identical to each other, unless $v_A = v_C$, $v_B = v_D$.

Now we can use the isomorphism $S\otimes S\cong\wedge^*\mathbb{R}^8$ specified by (A.19). We see that the action of any group element $g\in\mathrm{Spin}(8)$ on v_A, i.e. $v_A\to\tilde{v}_A\equiv(\rho^+\oplus\rho^-)(g)\circ v_A$, will induce two equivalent actions on $v_A\otimes v_B$ and on its image in $\wedge^*\mathbb{R}^8$. The first action is simply $(\rho^+\oplus\rho^-)\otimes(\rho^+\oplus\rho^-)(g) : v_A\otimes v_B\to\tilde{v}_A\otimes\tilde{v}_B$. The second action, when restricted to the components $T^{\mu_1\cdots\mu_p}\equiv\langle v_A,\Gamma^{\mu_1\cdots\mu_p}v_B\rangle\in\wedge^p\mathbb{R}^8$, is determined by $T^{\mu_1\cdots\mu_p}\to\tilde{T}^{\mu_1\cdots\mu_p}\equiv\langle\tilde{v}_A,\Gamma^{\mu_1\cdots\mu_p}\tilde{v}_B\rangle$, which in turn gives rise to the adjoint action $\Gamma^{\mu_1\cdots\mu_p}\to\rho(g)^T\Gamma^{\mu_1\cdots\mu_p}\rho(g) = \rho(g)^{-1}\Gamma^{\mu_1\cdots\mu_p}\rho(g)$, leading to the tensor representation $\wedge^p\rho_v$. It follows that

$$(\rho^+\oplus\rho^-)\otimes(\rho^+\oplus\rho^-)\cong\bigoplus_{p=0}^{8}\wedge^p\rho_v\cong 2\left(1\oplus\rho_v\oplus\wedge^2\rho_v\oplus\wedge^3\rho_v\right)\oplus\wedge^4\rho_v\,. \tag{A.20}$$

The isomorphism discussed above gives an identity known as the Fierz rearrangement formula. The vectors v_A, v_B in (A.19) can be replaced by arbitrary spinors $\phi = \phi^A v_A, \psi = \psi^A v_A \in S$. On the left hand side of this correspondence, we have the tensor product $\phi\otimes\psi$ with components $\phi^A\psi^B$, which can be viewed as a 16×16 matrix acting on S. The right hand side can also be considered as such a matrix if we replace the Clifford elements $e_{\mu_1}\wedge\cdots\wedge e_{\mu_p}$ by their γ-matrix representation $\Gamma_{\mu_1\cdots\mu_p}$. Since these two matrices are the same object, we must have

$$\phi^A\psi^B = \frac{1}{16}\sum_{p=0}^{8}\frac{1}{p!}(\phi^T\Gamma^{\mu_1\cdots\mu_p}\psi)\,\Gamma_{\mu_1\cdots\mu_p}{}^{AB} \tag{A.21}$$

here $(\Gamma_{\mu_1\cdots\mu_p})^{AB}$ denotes the matrix-element of $\Gamma_{\mu_1\cdots\mu_p}$ in the basis v_A. The coefficients $1/p!$ in (A.21) are introduced so as to ensure that the sum runs over each of the basis

[2] Clifford group $G_d\subset\mathrm{Cl}(d)$ in d-dimensions is a finite group whose generators can be presented by the abstract elements $\{e_1,\ldots,e_d,-1\}$ subject to the relation that -1 is central and that $(-1)^2 = 1, e_i^2 = -1$ and $e_ie_j = (-1)e_je_i$ for all $i\neq j$.

elements of $\wedge^*\mathbb{R}^8$ exactly once, and the factor 1/16 comes from a group-theoretical consideration, which is nothing but the inverse of the dimension of the irreducible representation for the Clifford group. That this factor must be equal to 1/16 may also be checked by taking the trace of (A.21): from (A.13) we recall that for odd p, the matrix $\Gamma_{\mu_1\cdots\mu_p}$ is always off-diagonal, thus having a vanishing trace. For even $p>0$, the cyclic property of the trace $\mathrm{Tr}(\Gamma_{\mu_1\mu_2\cdots\mu_p}) = \mathrm{Tr}(\Gamma_{\mu_2\cdots\mu_p\mu_1})$ together with the γ-matrix identity $\Gamma_{\mu_1\mu_2\cdots\mu_p} = -\Gamma_{\mu_2\cdots\mu_p\mu_1}$ also gives $\mathrm{Tr}(\Gamma_{\mu_1\mu_2\cdots\mu_p}) = 0$. So when taking the trace, only the first term ($p=0$) in the right hand side of (A.21) survives and it takes the value $\frac{1}{16}\phi^T\cdot\psi\,\mathrm{Tr}(I_{16\times16}) = \phi^T\cdot\psi$, which agrees exactly with the trace of the left hand side. As an aside, note that this kind of argument allows us to write down a trace formula for the γ-matrices: multiplying (A.21) by $\Gamma^{\nu_1\cdots\nu_q}{}_{AB} = (-1)^{[\frac{q+1}{2}]}\Gamma^{\nu_1\cdots\nu_q}{}_{BA}$, the left hand side becomes $\phi^T\Gamma^{\nu_1\cdots\nu_q}\psi$, while the right hand side is $\sum_{p=0}^{8}\frac{(-1)^{[\frac{q+1}{2}]}}{16p!}(\phi^T\Gamma^{\mu_1\cdots\mu_p}\psi)\,\mathrm{Tr}(\Gamma_{\mu_1\cdots\mu_p}\Gamma^{\nu_1\cdots\nu_q})$, and this gives

$$\mathrm{Tr}(\Gamma_{\mu_1\cdots\mu_p}\Gamma^{\nu_1\cdots\nu_q}) = 16p!(-1)^{[\frac{p+1}{2}]}\delta_{pq}\delta^{\nu_1}_{[\mu_1}\cdots\delta^{\nu_p}_{\mu_p]}. \tag{A.22}$$

For example we have $\mathrm{Tr}(\Gamma_\mu\Gamma^\nu) = -16\delta^\nu_\mu$, $\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\alpha\beta}) = -16(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu)$, etc.

We end this appendix with a few remarks. If $\phi=\psi$ has a definite chirality, then many terms in the sum (A.21) will vanish. Such terms correspond to $p\equiv 1,2$ (mod 4) when the γ-matrices are anti-symmetric or $p=$ odd when the γ-matrices map ϕ into some spinors with opposite chirality, which are orthogonal to ϕ^T. In this case the Fierz reaarangement formula gets much simplified:

$$\phi\phi^T = \frac{1}{16}\left((\phi^T\phi)I_{16\times16} + (\phi^T\Gamma_9\phi)\Gamma_9 + \frac{1}{4!}(\phi^T\Gamma^{\mu\nu\alpha\beta}\phi)\Gamma_{\mu\nu\alpha\beta}\right). \tag{A.23}$$

Thus, since $\Gamma_9\phi = \pm\phi$ for $\phi\in S^\pm$, we have

$$\phi\phi^T = \frac{1}{16}\left((\phi^T\phi)(I_{16\times16}\pm\Gamma_9) + \frac{1}{4!}(\phi^T\Gamma^{\mu\nu\alpha\beta}\phi)\Gamma_{\mu\nu\alpha\beta}\right), \qquad \phi\in S^\pm. \tag{A.24}$$

Clearly (A.24) defines a projector from S onto its one-dimensional subspace spaned by ϕ. Moreover, if $\phi\neq\psi$ but they still have the same definite chirality — say, both of them are in S^+ or in S^-, then by symmetrizing (A.21) we get

$$\phi^A\psi^B + \phi^B\psi^A = \frac{1}{8}\left((\phi^T\psi)(\delta^{AB}\pm{\Gamma_9}^{AB}) + \frac{1}{4!}(\phi^T\Gamma^{\mu\nu\alpha\beta}\psi){\Gamma_{\mu\nu\alpha\beta}}^{AB}\right), \tag{A.25}$$

where "$\pm$" corresponds to ϕ, $\psi\in S^\pm$, respectively. We can also anti-symmetrize (A.21) to derive, for ϕ and ψ having the same chirality,

$$\phi^A\psi^B - \phi^B\psi^A = \frac{1}{8}\left(\frac{1}{2!}(\phi^T\Gamma^{\mu\nu}\psi){\Gamma_{\mu\nu}}^{AB} + \frac{1}{6!}(\phi^T\Gamma^{\mu\nu\alpha\beta\lambda\rho}\psi){\Gamma_{\mu\nu\alpha\beta\lambda\rho}}^{AB}\right). \tag{A.26}$$

The two terms in the right hand side of (A.26) are in fact equal to each other upto a factor $\pm\Gamma_9$ and we finally have $\phi\psi^T - \psi\phi^T = \frac{1}{16}(\phi^T\Gamma^{\mu\nu}\psi)\Gamma_{\mu\nu}(1\pm\Gamma_9)$, if both ϕ, ψ are in $S^\pm$.

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